Muon transverse ionization cooling: Stochastic approach

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Transverse ionization cooling of muons is modeled as a Brownian motion of the muon beam as it traverses a Li or Be rod. A Langevin-like equation is written for the free particle case (no external transverse magnetic field) and for the case of a harmonically bound beam in the presence of a focusing magnetic field. We demonstrate that the well-known muon cooling equations for short absorbers can be extrapolated to the useful case of a long absorber rod with a focusing magnetic field present.

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I. INTRODUCTION

The possibility of a $\mu^+\mu^-$ collider to explore the Higgs energy range and supersymmetry has begun to be vigorously examined. One of the crucial issues to achieve the required luminosity ($\mathcal{L} \approx 10^{34}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$) is the need to compress the phase space by means of muon cooling. A technique that has been shown to be very promising is ionization cooling. The introduction of the concept and the physics was first discussed by Skrinsky [1]; for a clear and comprehensive treatment we refer the reader to Neuffer's articles [2].

The original derivation in Ref. [2] of the transverse cooling differential equation assumed a cooling system consisting of small alternating absorber and reaccelerator sections. Subsequently, Palmer [3] and Fernow [4] have argued that the equation is valid for a single long absorber. Their argument is quite straightforward; from the definition of emittance it is possible to show that

$$\frac{d\epsilon_\perp^N}{dz} = -\left|\frac{dE_\mu}{dz}\right| \frac{\epsilon_\perp^N}{\beta^2 E_\mu} + \frac{1}{2}\frac{(\gamma\beta)}{\epsilon_\perp^N(z)}\left[\langle r^2\rangle \frac{d\langle\theta^2\rangle}{dz} + \langle\theta^2\rangle\frac{d\langle r^2\rangle}{dz}\right] - \frac{(\gamma\beta)}{\epsilon_\perp^N(z)}\langle r\theta\rangle\frac{d\langle r\theta\rangle}{dz}, \quad (1)$$

where we have used $\frac{d(\gamma\beta)}{dz} = \frac{1}{\beta mc^2}\frac{dE_\mu}{dz}$, $\epsilon_\perp^N$ is the muons' normalized transverse emittance, E_μ is the muon total energy, $\beta_\perp$ is the beta function, and $\langle r^2\rangle$, $\langle\theta^2\rangle$ are the square of the rms position and divergence of the beam due to multiple scattering. The first term in Eq. (1) reflects the energy loss (*cooling*) and the last three terms are produced by multiple scattering (*heating*).

Assuming a long cooling rod (Li or Be) the Gaussian approximation is quite adequate; then

$$\langle y^2\rangle = \frac{1}{3}\theta_c^2 z^3,$$
$$\langle\theta^2\rangle = \theta_c^2 z, \quad (2)$$

where the projected scattering angle $\theta_0 \equiv \theta_c\sqrt{z} = \frac{13.6\ [\mathrm{MeV}]}{\beta cp}\sqrt{\frac{z}{L_R}}$ and L_R is the radiation length. In this expression we have neglected logarithmic correction terms [5]. There is also the more accurate formula for θ_0 [6],

$$\theta_0 = \frac{\chi_c}{\sqrt{1+F^2}}\sqrt{\frac{1+\nu}{\nu}\ln(1+\nu) - 1} \quad (3)$$

with the characteristic angle

$$\chi_c = \frac{\sqrt{0.157\ [\mathrm{MeV}]z}}{\beta p}\sqrt{\frac{Z(Z+1)}{A}};$$

the phenomenological parameters $F = 0.98$ and $\nu = \frac{\Omega_0}{2(1-F)}$ are chosen to fit the experimental data (Ω_0 represents the mean number of scatters in the medium).

If we neglect from Eq. (1) the third and fourth terms, then we can write the minimum achievable emittance as [2],

$$\epsilon_\perp^N|_{\min} = \frac{(13.6\ [\mathrm{MeV}])^2}{2\beta mc^2 |\frac{dE_\mu}{dz}|}\frac{\beta_\perp(0)}{L_R}. \quad (4)$$

Palmer [3] has pointed out that the third and fourth terms in Eq. (1) need not be included because the multiple scattering medium (Li or Be rod) is immersed in a uniform transverse magnetic field, which prevents the beam from spreading laterally. This raises the question, how is the functional form of the particle distribution changed in position and angle due to the external magnetic field? We will examine this question in the next sections.

We should also point out that the treatment considered here is also of interest for a number of other problems in accelerator physics. These include scattering of particles by residual gas in a synchrotron [7], scattering in the Inverse Čerenkov accelerators [8], plasma beat-wave accelerators [9], plasma lenses for future linear colliders [10], and the dynamics of space-charge dominated beams [11].

II. PARTICLE DISTRIBUTION WITHOUT A MAGNETIC FIELD

This problem has been analyzed in detail by Rossi [12] following the Fokker-Planck equation approach. Let

$\mathcal{W}(y,\theta,z;y_0,\theta_0)\,dy\,d\theta$ represent the number of particles in the phase space element $(y, y+dy; \theta, \theta+d\theta)$ after traversing a medium of thickness z and initial coordinates $y(0)=y_0$, $\theta(0)=\theta_0$. It satisfies the equation

$$\frac{\partial \mathcal{W}}{\partial z} = -(\theta-\Theta_0)\frac{\partial \mathcal{W}}{\partial y} + \frac{\theta_c^2}{2}\frac{\partial^2 \mathcal{W}}{\partial \theta^2} \tag{5}$$

with boundary conditions $\mathcal{W}(y,\theta,z;y_0,\theta_0)|_{z=0} = \delta(y-y_0)\delta(\theta-\theta_0)$ and solution

$$\mathcal{W}(y,\theta,z;y_0,\theta_0) = \frac{\sqrt{3}}{\pi z^2\theta_c^2}\exp\left[-\frac{2}{\theta_c^2}\left(\frac{(\theta-\Theta_0)^2}{z} - \frac{3(y-Y_0)(\theta-\Theta_0)}{z^2} + \frac{3(y-Y_0)^2}{z^3}\right)\right] \tag{6}$$

which can be verified by direct substitution with $\Theta_0=\theta_0$ and $Y_0 = y_0+\theta_0 z$. This result allows us to compute the emittance of the beam after traversing the cooling rod. After tedious Gaussian integrations we obtain

$$\begin{aligned}
\langle y\rangle &= Y_0,\\
\langle \theta\rangle &= \theta_0,\\
\langle y^2\rangle &= Y_0^2 + \frac{\theta_c^2 z^3}{3},\\
\langle \theta^2\rangle &= \Theta_0^2 + \theta_c^2 z,\\
\langle y\theta\rangle &= \Theta_0 Y_0 + \frac{\theta_c^2 z^2}{2}.
\end{aligned} \tag{7}$$

Averaging over the initial coordinates and assuming Gaussian distributions with standard deviations σ_{y0} and $\sigma_{\theta 0}$, we obtain

$$\begin{aligned}
\langle\langle y\rangle\rangle &= \langle\langle \theta\rangle\rangle = 0,\\
\langle\langle y^2\rangle\rangle &= \sigma_{y0}^2 + \sigma_{\theta 0}^2 z^2 + 2z\langle y_0\theta_0\rangle + \frac{\theta_c^2 z^3}{3},\\
\langle\langle \theta^2\rangle\rangle &= \sigma_{\theta 0}^2 + \theta_c^2 z,\\
\langle\langle y\theta\rangle\rangle &= \sigma_{\theta 0}^2 z + \langle y_0\theta_0\rangle + \frac{\theta_c^2 z^2}{2}.
\end{aligned} \tag{8}$$

The total emittance in the absence of a focusing field is

$$\epsilon_\perp(z) = \sqrt{\epsilon_\perp^2(0) + \frac{\theta_c^4 z^4}{12} + \sigma_{\theta 0}^2\theta_c^2\frac{z^3}{3} + \theta_c^2\langle y_0\theta_0\rangle z^2 + \sigma_{y0}^2\theta_c^2 z}. \tag{9}$$

The terms proportional to θ_c are the contributions due to multiple scattering.

III. PARTICLE DISTRIBUTION WITH A MAGNETIC FIELD

Consider now the problem of a particle traversing a rod of material that has an axial current flowing through it; such a particle satisfies the equation of motion $\frac{d^2y}{dz^2} + K(z)y = 0$, where $K(z) = \frac{eB}{mc\gamma\beta a} = \omega^2$, B is the azimuthal magnetic field and a is the radius of the rod. $K(z)$ is a function of z because of the energy loss, but for the simplicity of the arguments that follow, we neglect the energy change as the beam traverses the rod.

A more complete treatment must take into account random accelerations of the particles due to scattering (i.e., stochastic changes in angle $\frac{dy}{dz}=\theta$). A correct equation of motion is

$$\frac{dy}{dz} = \theta, \quad \frac{d\theta}{dz} + K(z)y = A(z), \tag{10}$$

where we denote with $A(z)$ the random acceleration due to Coulomb scattering which excites betatron oscillations in the beam. This equation is formally a Langevin equation of a particle in an external field $K(z)y$ (harmonic oscillator) where the frequency is a function of the *time* variable z. The main assumptions regarding the stochastic variable $A(z)$, more precisely $\int_z^{z+dz} dz' A(z')$, is that it is independent of y, that it varies extremely rapidly compared to the variations of the coordinates y and θ, and that it is Gaussian-distributed with a variance θ_c^2.

Therefore, casting the muon cooling problem in stochastic terms, we first determine the particle distribution $\mathcal{W}(y,\theta,z;y_0\theta_0)$, and then as before, we calculate the emittance from that function.

The method for determining the distribution function uses standard techniques for solving ordinary differential equations and is described in detail by Chandrasekhar [13]. The result for the distribution function is [14]

$$\mathcal{W}(y,\theta,z,\omega;y_0,\Theta_0) = \frac{1}{2\pi\sqrt{\{FG-H^2\}}}\exp\left[-\frac{1}{2\{FG-H^2\}}\left[G(y-Y_0)^2 - 2H(y-Y_0)(\theta-\Theta_0) + F(\theta-\Theta_0)^2\right]\right], \tag{11}$$

where the parameters F, G, and H are functions of the external focusing field;

$$
\begin{aligned}
F &= \theta_c^2 \frac{z}{2\omega^2}\left(1 - \frac{\sin 2\omega z}{2\omega z}\right), \\
G &= \theta_c^2 \frac{z}{2}\left(1 + \frac{\sin 2\omega z}{2\omega z}\right), \\
H &= \theta_c^2 \frac{1}{2\omega^2}\frac{1}{2}(1 - \cos 2\omega z) \\
FG - H^2 &= \frac{\theta_c^4}{\omega^4} z^2 \left[1 - \left(\frac{\sin \omega z}{\omega z}\right)^2\right]. \qquad (12)
\end{aligned}
$$

It can be shown that Eq. (11) reproduces Eq. (6) in the limit $\omega \rightarrow 0$, and the probability density $\mathcal{W}(y, \theta, z, \omega; y_0, \theta_0)$ satisfies a parabolic partial differential equation, the Fokker-Planck equation

$$
\frac{\partial \mathcal{W}}{\partial z} = -(\theta - \Theta_0)\frac{\partial \mathcal{W}}{\partial y} + \omega^2 (y - Y_0)\frac{\partial \mathcal{W}}{\partial \theta} + \frac{1}{2}\theta_c^2 \frac{\partial^2 \mathcal{W}}{\partial \theta^2}. \qquad (13)
$$

As in the preceding section we are interested in calculating the second moments of the distribution and from those the emittance. We find that

$$
\begin{aligned}
\langle y \rangle &= y_0 \cos \omega z + \theta_0 z \frac{\sin \omega z}{\omega z}, \\
\langle \theta \rangle &= \theta_0 \cos \omega z - y_0 \omega \sin \omega z, \\
\langle y^2 \rangle &= \left(y_0 \cos \omega z + \theta_0 z \frac{\sin \omega z}{\omega z}\right)^2 \\
&\quad + z \frac{\theta_c^2}{2\omega^2}\left(1 - \frac{\sin 2\omega z}{2\omega z}\right), \\
\langle \theta^2 \rangle &= (\theta_0 \cos \omega z - y_0 \omega \sin \omega z)^2 \\
&\quad + z \frac{\theta_c^2}{2}\left(1 + \frac{\sin 2\omega z}{2\omega z}\right), \\
\langle y\theta \rangle &= y_0 \theta_0 \cos 2\omega z - \frac{y_0^2 \omega}{2} \sin 2\omega z \\
&\quad + \theta_0^2 z \cos \omega z \frac{\sin \omega z}{\omega z} + \frac{\theta_c^2 z^2}{2}\left(\frac{\sin \omega z}{\omega z}\right).^2 \qquad (14)
\end{aligned}
$$

If we now assume an uncorrelated ensemble of incident particles with independent Gaussian distributions of initial conditions y_0 and θ_0 and average over both variables, we obtain

$$
\begin{aligned}
\langle\langle y \rangle\rangle &= \langle\langle \theta \rangle\rangle = 0, \\
\langle\langle y^2 \rangle\rangle &= \sigma_{y0}^2 (\cos \omega z)^2 + \sigma_{\theta 0}^2 z^2 \left(\frac{\sin \omega z}{\omega z}\right)^2 \\
&\quad + z \frac{\theta_c^2}{2\omega^2}\left(1 - \frac{\sin 2\omega z}{2\omega z}\right), \\
\langle\langle \theta^2 \rangle\rangle &= \sigma_{\theta 0}^2 (\cos \omega z)^2 + \sigma_{y0}^2 \omega^2 (\sin \omega z)^2 \\
&\quad + z \frac{\theta_c^2}{2}\left(1 + \frac{\sin 2\omega z}{2\omega z}\right), \\
\langle\langle y\theta \rangle\rangle &= \sigma_{\theta 0}^2 z \frac{\sin 2\omega z}{2\omega z} - \sigma_{y0}^2 \frac{1}{2}\omega \sin 2\omega z \\
&\quad + \frac{\theta_c^2 z^2}{2}\left(\frac{\sin \omega z}{\omega z}\right)^2. \qquad (15)
\end{aligned}
$$

An important observation is that for a very high magnetic field ($\omega L_{\rm rod} \gg 1$) the rms beam size remains approximately constant. If the beam is focused to a waist at the entrance to the rod, the emittance at any distance z inside the rod is [15]

$$
\epsilon_\perp^2(z) = \epsilon_\perp^2(0) + \sigma_{y0}^2 \theta_c^2 z + \frac{\theta_c^4}{4\omega^2} z^2 + \frac{\theta_c^4}{8\omega^4}\left[1 - \cos(2\omega z)\right]. \qquad (16)
$$

The second term gives the dominant contribution of multiple scattering to the emittance. This also leads to Eq. (4) for the minimum emittance, provided that $\sigma_{y0}^2 \gg \frac{\theta_c^2}{2\omega^2} L_{\rm rod}$ and $\sigma_{y0}^2 \gg \frac{\theta_c^2}{4\omega^3}$, which will usually be the case for strong focusing.

IV. CONCLUSIONS

Using an analogy with a random dynamical process modeled with a Langevin equation, we have incorporated the stochastic nature of both position and angle variables into the problem of a muon traversing a Li or Be rod immersed in a uniform azimuthal magnetic field. The pseudo-Brownian motion of the particles in the medium represents heating of the beam. The emittance increase due to multiple Coulomb scattering (the less likely single and plural scattering events are neglected) opposes the emittance decrease (cooling), introduced by the energy loss $\frac{dE_\mu}{dz}$. With sufficiently strong focusing present, ionization cooling can effectively take place over long lengths of absorbing material.

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